Core 3 - January 2008

(1) a) i) $y = (2x^2 - 5x + 1)^{20} \rightarrow \frac{dy}{dx} = 20(4x - 5)(2x^2 - 5x + 1)^{19}$

ii) $y = x\cos(x)$

$u = x \qquad v = \cos(x)$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = -\sin(x)$

$\frac{dy}{dx} = \cos(x) - x\sin(x)$

Product rule!

b) $y = \frac{x^3}{x-2}$

$u = x^3 \qquad v = x - 2$

$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = 1$

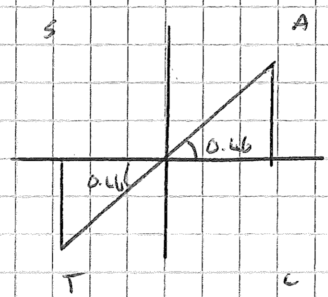
$\frac{dy}{dx} = \frac{3x^2(x-2) - x^3}{(x-2)^2}$

$= \frac{3x^3 - 6x^2 - x^3}{(x-2)^2} = \frac{2x^3 - 6x^2}{(x-2)^2} = \frac{2x^2(x-3)}{(x-2)^2}$

(2) a) $\cot(x) = 2 \rightarrow \tan(x) = 1/2$

$\rightarrow x = 0.4636...$

$x = 0.46,\ 3.61$



b) $\operatorname{cosec}^2(x) = \frac{3\cot(x) + 4}{2}$

$1 + \cot^2(x) = \frac{3\cot(x) + 4}{2}$

$2 + 2\cot^2(x) = 3\cot(x) + 4$

$2\cot^2(x) - 3\cot(x) - 2 = 0$

c) $(2\cot(x) + 1)(\cot(x) - 2) = 0$

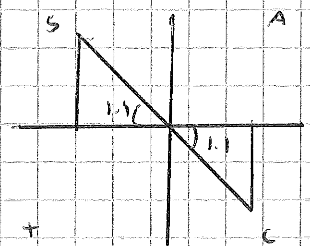
$2\cot(x) + 1 = 0$

$\cot(x) = -1/2$

$\tan(x) = -2 \qquad [x = -1.107...]$

$\cot(x) = 2$

$x = 0.46,\ 3.61$ (from a)



$x = 2.034...,\ 5.176...$

Final answer: $x = 0.46,\ 3.61,\ 2.03,\ 5.18$

③ a) Let $f(x) = x + (1+3x)^{1/4}$

$f(-0.33) = -0.33 + (1+3(-0.33))^{1/4} = -0.01377.$

$f(-0.32) = -0.32 + (1+3(-0.32))^{1/4} = 0.1272.$

change of sign, ∴ root lies between

b) $x + (1+3x)^{1/4} = 0$

$x = -(1+3x)^{1/4}$

$x^4 = (1+3x)$

raise both sides to power of 4 → both positive

$x^4 = 1 + 3x$

$3x = x^4 - 1 \rightarrow x = \frac{1}{3}(x^4 - 1)$

c) $x_1 = -0.3$

$x_2 = \frac{1}{3}(-0.3^4 - 1) = -0.3306$

$x_3 = \frac{1}{3}(-0.33..^4 - 1) = -0.3293..$

$x_4 = \frac{1}{3}(-0.329^4 - 1) = -0.3294.. = -0.329$ (3sf)

④ a) Range: $f(x) \in \mathbb{R}$ all real numbers

b) i) $fg(x) = f\left(\frac{1}{x-3}\right) = \left(\frac{1}{x-3}\right)^3$

ii) $\left(\frac{1}{x-3}\right)^3 = 64 \rightarrow \frac{1}{x-3} = 4$

$1 = 4(x-3)$

$x-3 = \frac{1}{4} \rightarrow x = 3\frac{1}{4}$

c) i) let $y = \frac{1}{x-3}$

$y(x-3) = 1 \rightarrow x-3 = \frac{1}{y}$

$\rightarrow x = \frac{1}{y} + 3$

$\rightarrow y = \frac{1}{x} + 3 = g^{-1}(x)$

ii) Range $g^{-1}(x)$ = Domain of $g(x)$

$\rightarrow$ Range $\neq 3$

⑤ a) i) $y = 2x^2 - 8x + 3 \rightarrow dy/dx = 4x - 8$

ii) $\int \frac{x-2}{2x^2-8x+3}\,dx = \frac{1}{4}\int \frac{4x-8}{2x^2-8x+3}\,dx$

$= \left[\frac{1}{4}\ln(2x^2-8x+3)\right]_4^6 = \frac{1}{4}\ln(2\times 6^2 - 8(6) + 3) - \frac{1}{4}\ln(2\times 4^2 - 8(4) + 3)$

$= \frac{1}{4}\ln(27) - \frac{1}{4}\ln(3)$

$= \frac{1}{4}\ln(27/3) = \frac{1}{4}\ln(9) = \frac{1}{4}\ln(3^2)$

$= \frac{2}{4}\ln(3) = \frac{1}{2}\ln(3)$

b) $\int x\sqrt{3x-1}\,dx$

$u = 3x-1$

$du/dx = 3 \rightarrow dx = \frac{1}{3}\,du$

$\int x\,u^{1/2}\,\frac{1}{3}\,du$

$u = 3x-1$

$u+1 = 3x$

$x = \frac{1}{3}(u+1)$

$\frac{1}{3}\int (u+1)u^{1/2}\,\frac{1}{3}\,du$

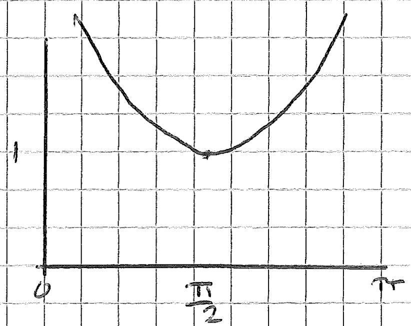
$= \frac{1}{9}\int u^{3/2} + u^{1/2}\,du$

$= \frac{1}{9}\left[\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right]$

$= \frac{1}{9}\left[\frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3}\right]$

$= \frac{2u^{5/2}}{45} + \frac{2u^{3/2}}{27} = \frac{2}{45}(3x-1)^{5/2} + \frac{2}{27}(3x-1)^{3/2}$

⑥ a)



b)

x	y
0.15	6.692
0.25	4.042
0.35	2.916
0.45	2.299

$h = 0.1$

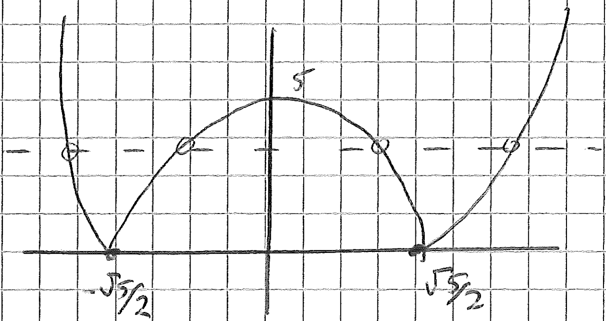
$A = 0.1 \times 15.949$

$= 1.59$ (3sf)

7) a) $\boxed{1st}$ $4x$ → stretch y direction scale factor 4

$\boxed{2nd}$ -5 → translation $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$

b)



$y = 4x^2 - 5$

crosses x when $y = 0$

$\rightarrow$ $0 = 4x^2 - 5$

$5 = 4x^2$

$x = \pm\sqrt{5/4}$

$= \pm \sqrt{5}/2$

c) i) From graph above:

$4x^2 - 5 = 4$
$4x^2 = 9$
$x^2 = 9/4$
$x = \pm 3/2$

$\boxed{AND}$

$-4x^2 + 5 = 4$
$4x^2 = 1$
$x^2 = 1/4$
$x = \pm 1/2$

Solutions: $x = -3/2, \ -1/2, \ 1/2, \ 3/2$

ii) From graph: $x \geq 3/2$, $x \leq -3/2$

$-1/2 \leq x \leq 1/2$

8) a) $e^{-2x} = 3$

$-2x = \ln(3)$

$x = -1/2 \ln(3)$

b) $\int x e^{-2x} dx$

$u = x$ $\quad dv/dx = e^{-2x}$

$du/dx = 1$ $\quad v = -1/2 e^{-2x}$

$= uv - \int v \, du/dx$

$= -1/2 x e^{-2x} - \int -1/2 e^{-2x}$

$= -1/2 x e^{-2x} + 1/2 \int e^{-2x}$

$= -1/2 x e^{-2x} - 1/4 e^{-2x} + C$

c) i) $y = e^{-2x} + 6x$

At st point, $dy/dx = 0$

$$dy/dx = -2e^{-2x} + 6 = 0$$

$$e^{-2x} - 3 = 0$$

$$e^{-2x} = 3$$

$$-2x = \ln(3)$$

$$x = -1/2 \ln(3)$$

$$y = e^{-2(-1/2 \ln(3))} + 6(-1/2 \ln(3))$$

$$= e^{\ln(3)} - 3\ln(3)$$

$$= 3 - 3\ln(3)$$

$\therefore$ co-ordinates $= (-1/2 \ln(3), 3 - 3\ln(3))$

ii) $d^2y/dx^2 = 4e^{-2x}$

$$x = -1/2 \ln(3) \rightarrow 4e^{-2(-1/2 \ln(3))}$$

$$= 4e^{\ln(3)} = 4 \times 3 = 12$$

$12 > 0 \therefore$ minimum point.

iii) $V = \pi \int y^2 \, dx$

$$y = e^{-2x} + 6x$$

$$y^2 = (e^{-2x} + 6x)(e^{-2x} + 6x)$$

$$= e^{-4x} + 12xe^{-2x} + 36x^2$$

$$V = \pi \int e^{-4x} + \boxed{12xe^{-2x}} + 36x^2$$

$$\boxed{12x\, e^{-2x}}$$

Use Integration by parts:

~~answer~~ From b)

$$\int xe^{-2x} = -1/2 xe^{-2x} - 1/4 e^{-2x}$$

so $\int 12xe^{-2x} = -6xe^{-2x} - 3e^{-2x}$

$$= \pi \left[-1/4 e^{-4x} - 6xe^{-2x} - 3e^{-2x} + 12x^3 \right]_0^1$$

$$= \pi \left[-1/4 e^{-4} - 6e^{-2} - 3e^{-2} + 12 \right]$$

$$- \pi \left[-1/4 e^0 - 0 - 3e^0 + 0 \right]$$

$$= \pi \left[15 1/4 - 9e^{-2} - 1/4 e^{-4} \right]$$

$$= 44.1 \text{ (3sf)}$$